MATHEMATICS

A MULTIDIMENSIONAL GENERALIZATION OF THE PADÉ TABLE. I

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INTRODUCTION

The initial ideas of the investigations of this paper can be found in HERMITE's famous work of 1873 "Sur la fonction exponentielle" [3] in which for the first time the transcendence of the number e was proved. HERMITE's fundamental principle there is the following [1]).

Let $\varrho_1, \varrho_2, \ldots, \varrho_m$ be m arbitrarily chosen positive integers, $\sigma=\sum_{\mu=1}^{m} \varrho_\mu$, and let $\omega_1, \omega_2, \ldots, \omega_m$ be a set of m different real or complex numbers. Then it is possible to construct a set of m polynomials

$$(1) \qquad (\mathfrak{a}_1(x), \mathfrak{a}_2(x), \ldots, \mathfrak{a}_m(x))$$

such that

a) the degree of each $\mathfrak{a}_\mu(x)$ is $\sigma-\varrho_\mu$, $\mu=1, 2, \ldots, m$,

b) in each of the power series $\mathfrak{r}_{kl}(x)=e^{\omega_k x}\,\mathfrak{a}_l(x)-e^{\omega_l x}\,\mathfrak{a}_k(x)$, $k, l=1, 2, \ldots, m$, $k\neq l$, the first $\sigma+1$ terms vanish, i.e. the "order" of each series $\mathfrak{r}_{kl}(x)$ in $x=0$ is equal to $\sigma+1$.

Apparently such "algebraic approximations" of the exponential function (and also of others, see e.g. [1]) occupied HERMITE greatly. In a letter of 1873 to BORCHARDT [2] he introduced a different but related type of approximations. In it occurs a set of m polynomials

$$(2) \qquad (a_1(x), a_2(x), \ldots, a_m(x))$$

such that

a) the degree of $a_\mu(x)$ is $\varrho_\mu-1$, $\mu=1, 2, \ldots, m$,

b) the order in $x=0$ of the remainder function $r(x)=\sum_{\mu=1}^{m} e^{\omega_\mu x}\, a_\mu(x)$ is equal to $\sigma-1$.

This function $r(x)$ is expressed as a repeated integral. Twenty years later HERMITE returned to this subject in a letter to PINCHERLE [4]. On this occasion he observed that the polynomials $a_\mu(x)$ can be given by the formula

$$a_\mu(x)=\operatorname*{Res}_{z=\omega_\mu}\; e^{zx} \prod_{\mu=1}^{m} (z-\omega_\mu)^{-\varrho_\mu}, \quad \mu=1, 2, \ldots, m.$$

For a proof see e.g. our section 4.2 [2]).

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[1]) We use a different notation.

[2]) The first number refers to the chapter, the second to the section of the chapter.

HERMITE's method was rediscovered by MAHLER. In the important paper of 1932 "Zur Approximation der Exponentialfunktion und des Logarithmus" [6] MAHLER used the polynomials $a_\mu(x)$ extensively. Mahler also proved there the following remarkable relation between sets of polynomials (1) and sets of polynomials (2):

Since the polynomials $\mathfrak{a}_\mu(x)$ and $a_\mu(x)$ depend on the choice of the m positive integers $\varrho_1, \varrho_2, \ldots, \varrho_m$, let us write $\mathfrak{a}_\mu(x/\varrho_1, \varrho_2, \ldots, \varrho_m)$ and $a_\mu(x/\varrho_1, \varrho_2, \ldots, \varrho_m)$ instead of $\mathfrak{a}_\mu(x)$ and $a_\mu(x)$, $\mu = 1, 2, \ldots, m$. Let $\delta_{\mu\nu}$ denote Kronecker's symbol, and let the polynomials $\mathfrak{a}_{\mu\nu}(x/\varrho_1, \varrho_2, \ldots, \varrho_m)$ and $a_{\mu\nu}(x/\varrho_1, \varrho_2, \ldots, \varrho_m)$ be defined by

$$\mathfrak{a}_{\mu\nu}(x/\varrho_1, \varrho_2, \ldots, \varrho_m) = \mathfrak{a}_\nu(x/\varrho_1 - \delta_{\mu 1}, \varrho_2 - \delta_{\mu 2}, \ldots, \varrho_m - \delta_{\mu m}) \text{ [3]},$$
$$a_{\mu\nu}(x/\varrho_1, \varrho_2, \ldots, \varrho_m) = a_\nu(x/\varrho_1 + \delta_{\mu 1}, \varrho_2 + \delta_{\mu 2}, \ldots, \varrho_m + \delta_{\mu m}).$$

Construct with these two sets of m^2 polynomials the two matrices

$$\mathfrak{A}(x/\varrho_1, \varrho_2, \ldots, \varrho_m) = (\mathfrak{a}_{\mu\nu}(x/\varrho_1, \varrho_2, \ldots, \varrho_m))_{\substack{\mu=1,2,\ldots,m\\ \nu=1,2,\ldots,m}}$$

and

$$A(x/\varrho_1, \varrho_2, \ldots, \varrho_m) = (a_{\mu\nu}(x/\varrho_1, \varrho_2, \ldots, \varrho_m))_{\substack{\mu=1,2,\ldots,m\\ \nu=1,2,\ldots,m}}.$$

Then MAHLER finds the relation

$$(3) \qquad A(x/\varrho_1, \varrho_2, \ldots, \varrho_m)\, \mathfrak{A}^T(x/\varrho_1, \varrho_2, \ldots, \varrho_m) = \begin{pmatrix} c_0 x^\sigma & & & \\ & c_1 x^\sigma & & 0 \\ & 0 & \ddots & \\ & & & c_m x^\sigma \end{pmatrix}$$

where $c_1, c_2, \ldots, c_m$ are constants.

MAHLER also proved that the two polynomials $\det(\mathfrak{A}(x/\varrho_1, \varrho_2, \ldots, \varrho_m))$ and $\det(A(x/\varrho_1, \varrho_2, \ldots, \varrho_m))$ take very simple forms viz.:

$$(4) \qquad \begin{cases} \det(\mathfrak{A}(x/\varrho_1, \varrho_2, \ldots, \varrho_m)) = a x^{(m-1)\sigma} \\ \det(A(x/\varrho_1, \varrho_2, \ldots, \varrho_m)) = b x^\sigma \end{cases}$$

where a and b are constants.

Later MAHLER developed a much more general theory. Then he started from m formal power series $f_1(x), f_2(x), \ldots, f_m(x)$ instead of from the special functions $e^{\omega_1 x}, e^{\omega_2 x}, \ldots, e^{\omega_m x}$.

It is not difficult to show that also in this case there exists for m positive integers $\varrho_1, \varrho_2, \ldots, \varrho_m$ a non-trivial set of polynomials $a_1(x), a_2(x), \ldots, a_m(x)$ such that

a) the degree of $a_\mu(x)$ is at most $\varrho_\mu - 1$, $\mu = 1, 2, \ldots, m$,

[3]) $\varrho_\nu - \delta_{\mu\nu}$ can be zero, $\nu, \mu = 1, 2, \ldots, m$. But since the polynomials (1) can also be defined in case that one or more or the numbers ϱ_μ are zero, this does not matter here.

b) the order in $x=0$ of the remainder function $r(x)=\sum_{\mu=1}^{m} a_\mu(x)\, f_\mu(x)$ is at least $\sigma-1$, $\sigma=\sum_{\mu=1}^{m}\varrho_\mu$;

compare our theorem 1.1.1, part A.

On the other hand, there exists for every set of m non-negative integers $\varrho_1, \varrho_2, \ldots, \varrho_m$ a non-trivial set of polynomials $\mathfrak{a}_1(x), \mathfrak{a}_2(x), \ldots, \mathfrak{a}_m(x)$ such that

a) the degree of $\mathfrak{a}_\mu(x)$ is at most $\sigma-\varrho_\mu$, $\mu=1, 2, \ldots, m$,

b) the order in $x=0$ of the power series $\mathfrak{r}_{kl}(x)=\mathfrak{a}_l(x)\, f_k(x)-\mathfrak{a}_k(x)\, f_l(x)$, $k, l=1, 2, \ldots, m$, is at least $\sigma+1$;

compare our theorem 1.1.1, part B.

Mahler proved that if the power series are only subject to the condition that at least one of them has a non-zero constant term, the relations (3) and (4) still hold. Our theorems 2.1.1, 2.1.2 and 2.1.3 contain these results as special cases.

Mahler put the unpublished manuscript at the disposal of the author of the present paper and gave his permission for the free use of its results. For this kindness I wish to express my deepest gratitude to Professor Mahler.

The aim of the following investigations is:

a) To generalize the above outlined theory as far as possible. This is mainly done in the first two chapters.

b) To study special choices of the power series $f_1(x), f_2(x), \ldots, f_m(x)$ more thoroughly.

c) To bring the results of b) into connection with well-known results in the theory of the Padé table.

In Hermite's case of a system of exponential functions $e^{\omega_1 x}, e^{\omega_2 x}, \ldots, e^{\omega_m x}$, the degree of the polynomial $a_\mu(x)$ is always exactly $\varrho_\mu-1$, $\mu=1, 2, \ldots, m$, the order of the remainder function $r(x)$ is exactly $\sigma-1$, and for a fixed choice of positive integers $\varrho_1, \varrho_2, \ldots, \varrho_m$ the corresponding system of polynomials $a_1(x), a_2(x), \ldots, a_m(x)$ is uniquely determined apart, of course, from a multiplicative non-zero constant. Moreover, for the polynomials $\mathfrak{a}_1(x), \mathfrak{a}_2(x), \ldots, \mathfrak{a}_m(x)$ similar results hold. In the general case of m arbitrary power series $f_1(x), f_2(x), \ldots, f_m(x)$ these assertions need not be true. In this connection Mahler introduced the notion of *a perfect system*. Our own definition differs somewhat from Mahler's. First we introduce the notion of *normality*:

Let $f_1(x), f_2(x), \ldots, f_m(x)$ be m formal power series, let $\varrho_1, \varrho_2, \ldots, \varrho_m$ be m positive integers, and let $(a_1(x), a_2(x), \ldots, a_m(x))$ be a set of m polynomials such that

a) the degree of $a_\mu(x)$ is at most $\varrho_\mu-1$, $\mu=1, 2, \ldots, m$,

b) the order in $x=0$ of $r(x)=\sum_{\mu=1}^{m} a_\mu(x) f_\mu(x)$ is at least $\sigma-1$.

Then the system of m power series $(f_1(x), f_2(x), \ldots, f_m(x))$ is said to be normal for the set of positive integers $(\varrho_1, \varrho_2, \ldots, \varrho_m)$ if the order of $r(x)$ is necessarily exactly $\sigma-1$, for every set of m polynomials $(a_1(x), a_2(x), \ldots$ $\ldots, a_m(x))$ satisfying a) and b), definition 1.1.1. The system of power series is called *perfect* if it is normal for every set of m positive integers and if, moreover, the same is true for every subset of n power series, $2 \leqslant n < m$, taken from $(f_1(x), f_2(x), \ldots, f_m(x))$, definition 1.1.2.

If a system of power series is perfect, then

I the polynomials $a_\mu(x/\varrho_1, \varrho_2, \ldots, \varrho_m)$ and $\mathfrak{a}_\mu(x/\varrho_1, \varrho_2, \ldots, \varrho_m)$ belonging to it actually assume their highest possible degrees namely $\varrho_\mu - 1$ in the first and $\sigma - \varrho_\mu$ in the second case, $\mu = 1, 2, \ldots, m$, compare theorem 1.1.4 and theorem 2.2.1, part A,

II for every set of m non-negative integers $\varrho_1, \varrho_2, \ldots, \varrho_m$ at least one of the remainder series $\mathfrak{r}_{kl}(x) = \mathfrak{a}_l(x)f_k(x) - \mathfrak{a}_k(x)f_l(x)$ actually assumes its lowest possible order $\sigma+1$, compare theorem 2.2.1, part C,

III the polynomial systems $(a_1(x), a_2(x), \ldots, a_m(x))$ and $(\mathfrak{a}_1(x), \mathfrak{a}_2(x), \ldots$ $\ldots, \mathfrak{a}_m(x))$ are uniquely determined except for a multiplicative non-zero constant, compare theorem 1.1.2 and theorem 2.2.1, part B.

The system of functions $e^{\omega_1 x}, e^{\omega_2 x}, \ldots, e^{\omega_m x}$, $\omega_i \neq \omega_j$, $i \neq j$, $i, j = 1, 2, \ldots, m$, considered by HERMITE offers a first example of a perfect system (theorem 1.2.1). In chapter IV we discuss this particular system in detail.

A second example of a perfect system is formed by the binomial functions $(1-x)^{\omega_1}, (1-x)^{\omega_2}, \ldots, (1-x)^{\omega_m}$, under the conditions $\omega_i - \omega_j \neq 0, \pm 1, \pm 2, \ldots$, $i \neq j$, $i, j = 1, 2, \ldots, m$, (theorem 1.2.2). MAHLER [5] has already given explicit expressions for the polynomials $a_\mu(x)$ belonging to such a system. In section 3.1 we show that these polynomials are generalized hypergeometric polynomials with argument $1-x$ instead of x.

In the same section we show that the remainder function

$$r(x) = \sum_{\mu=1}^{m} a_\mu(x)(1-x)^{\omega_\mu}$$

reduces to a special case of Meijer's G-function, also with argument $1-x$ instead of x.

In section 3.3 we derive expressions for the polynomials $\mathfrak{a}_\mu(x)$ connected with this binomial function system. These expressions and their derivations are very similar to those of HERMITE's in [3] for the exponential function system $e^{\omega_1 x}, e^{\omega_2 x}, \ldots, e^{\omega_m x}$. Briefly, one can say that in HERMITE's proof differentiation is replaced by the application of the difference operator with difference 1, and integration by some sort of summation.

It seems to be very difficult to find more examples of perfect systems. The following system of power series

$$(\log^{m-1}(1-x),\ \log^{m-2}(1-x),\ \ldots,\ 1)$$

however is "partly perfect", for we show that this system is normal for every set of positive integers $\varrho_1, \varrho_2, \ldots, \varrho_m$ such that $\varrho_1 \leqslant \varrho_2 \leqslant \ldots \leqslant \varrho_m$, theorem 1.2.3. In order to study such systems we can use the same kind of arguments as in the case of a perfect system, but in detail the reasoning must be refined much more.

This is done in section 2.3. For this case we find that

I if $\varrho_1 \leqslant \varrho_2 \leqslant \ldots \leqslant \varrho_m$, the polynomials $a_\mu(x/\varrho_1, \varrho_2, \ldots, \varrho_m)$ and $\mathfrak{a}_\mu(x/\varrho_1, \varrho_2, \ldots, \varrho_m)$ are uniquely determined apart from a multiplicative non-zero constant, $\mu = 1, 2, \ldots, m$, theorem 1.1.2 and theorem 2.3.2, part A,

II if $\varrho_1 < \varrho_2 < \ldots < \varrho_m$, the polynomials $a_\mu(x/\varrho_1, \varrho_2, \ldots, \varrho_m)$ and $\mathfrak{a}_\mu(x/\varrho_1, \varrho_2, \ldots, \varrho_m)$ actually assume their highest possible degrees $\varrho_\mu - 1$ and $\sigma - \varrho_\mu$ respectively, $\mu = 1, 2, \ldots, m$, theorem 1.1.3 and theorem 2.3.2, part C,

III if $\varrho_1 \leqslant \varrho_2 \leqslant \ldots \leqslant \varrho_m$, at least one of the remainder functions $\mathfrak{r}_{kl}(x) = \mathfrak{a}_l(x) \log^{m-k}(1-x) - \mathfrak{a}_k(x) \log^{m-l}(1-x)$ assumes its lowest possible order $\sigma + 1$, $k, l = 1, 2, \ldots, m$, $k \neq l$, theorem 2.3.2, part B.

In section 5.1 we show that the polynomial $a_1(x/\varrho_1, \varrho_2, \ldots, \varrho_m)$ is a generalized hypergeometric polynomial with argument $1-x$ and the remainder function $r(x) = \sum_{\mu=1}^{m} a_\mu(x) \log^{m-\mu}(1-x)$ a special case of Meijer's G-function also with argument $1-x$, if $\varrho_1 \leqslant \varrho_2 \leqslant \ldots \leqslant \varrho_m$.

For $m=2$, $f_1(x)=1$ and $f_2(x)$ a power series with a non-zero constant term, the whole theory reduces to the study of the Padé table [4]).

We show that the classical results of PADÉ [7], [8] on the Padé fractions and remainder functions of $(1-x)^\alpha$, $\alpha \neq 0, \pm 1, \pm 2, \ldots$ and $\log(1-x)/x$ are contained in our much more general results. Moreover, the way at which we arrive at these results of PADÉ seems to be new.

[4]) For a short and clear introduction to the theory of the Padé table the reader is referred to PERRON [9] or WALL [10].

(To be continued)

REFERENCES

1. HERMITE, C., Sur l'expression U sin x+V cos x+W, extrait d'une lettre à M. Paul Gordan, 1873, Oeuvres tome III, 135–145.
2. ———, Sur quelques approximations algébriques, extrait d'une lettre à M. Borchardt, 1873, Oeuvres tome III, 146–149.
3. ———, Sur la fonction exponentielle, 1873, Oeuvres tome III, 150–181.
4. ———, Sur la généralisation des fractions continues algébriques, extrait d'une lettre à M. Pincherle, 1893, Oeuvres tome IV, 357–377.
5. MAHLER, K., Ein Beweis des Thue-Siegelschen Satzes über die Approximation algebraischer Zahlen für binomische Gleichungen, Mathematische Annalen, **105**, 267–276 (1931).
6. ———, Zur Approximation der Exponentialfunktion und des Logarithmus, Teil I, Journ. f.d.r.u.a. Math. **166**, 118–136 (1932).

7. Padé, H., Sur l'expression générale de la fraction rationnelle approchée de $(1+x)^m$, Comptes Rendus de l'Académie des Sciences **132**, 754–756 (1901).
8. ———, Recherches sur la convergence des développements en fractions continues d'une certaine catégorie de fonctions, Annales Scientifiques de l'Ecole Normale Supérieure XXIV, 341–400 (1907).
9. Perron, O., Die Lehre von den Kettenbrüchen II, Stuttgart 1957.
10. Wall, H. S., The analytic theory of Continued Fractions, Toronto, New York, London, 1948.